About Some Problems of Spectral Synthesis

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ABSTRACT

For a given pair $(A, b) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times 1}$ such that A is cyclic and b is a cyclic generator (with respect to A) of $\mathbb{R}^{n\times 1}$, it is shown that for every nonnegative integer m we can find a nonnegative integer t and a sequence $(f_j)_{j=0}^t$, $f_j \in \mathbb{R}^{1\times n}$, so that all the zeros of the rational function det $P(z)$, where $P(z) = zI - A - \sum_{j=0}^t z^{-(m+j)} b f_j$, lie in the open unit disc in the complex plane. The result is directly applicable to a stabilizability problem for linear systems with a time delay in control action.

1. INTRODUCTION AND PROBLEM STATEMENT

Let a pair $(A, b) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times 1}$ be given, A be *cyclic*, and b be a *cyclic generator* (with respect to A) of $\mathbb{R}^{n\times 1}$, i.e., $\operatorname{rank}[b, Ab, \ldots, A^{n-1}b] = n$. The problem considered is as follows:

PROBLEM A. For a given $m \in \mathbb{N}_0 \triangleq \mathbb{N} \cup \{0\}$, find $t \in \mathbb{N}_0$ and a sequence $(f_j)_{j=0}^t$, $f_j \in \mathbb{R}^{1\times n}$, so that all the zeros of the rational function

$$p(z) \triangleq \det\left[zI - A - \sum_{j=0}^{t} z^{-(m+j)} b f_j \right]$$

lie in the open unit disc in the complex plane.

The problem defined above is motivated by a (feedback) stabilizability question which is described in the last section of the paper.

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Without loss of generality we shall assume that

$$A=\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ a_1 & a_2 & a_3 & \cdots & a_n \end{bmatrix}, \qquad b=\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

Let $P(z)\triangleq zI-A-\sum_{j=0}^{t} z^{-(m+j)}bf_j$ and $f_j\triangleq [f_{j1},f_{j2},\dots,f_{jn}]$, $j=0,1,\dots,t$. Let $\tilde{a}_k\in\mathbb{R}[z^{-1}]$, where $k=1,2,\dots,n$, be given by the formula

$$\tilde{a}_k=\tilde{a}_k(z^{-1})\triangleq a_k+\sum_{j=0}^{t}(z^{-1})^{m+j}f_{jk}.$$

We have

$$P(z)=\begin{bmatrix} z & -1 & 0 & \cdots & 0 & 0 \\ 0 & z & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & z & -1 \\ -\tilde{a}_1 & -\tilde{a}_2 & -\tilde{a}_3 & \cdots & -\tilde{a}_{n-1} & z-\tilde{a}_n \end{bmatrix}.$$

Standard calculations justify the following equality:

$$p(z)=z^n-\left[\tilde{a}_1(z^{-1})+z\tilde{a}_2(z^{-1})+\cdots+z^{n-1}\tilde{a}_n(z^{-1})\right],$$

where $p(z)\triangleq\det P(z)$ is a rational function of z. Observe that $\bar{p}(z)\triangleq z^{m+t}p(z)$ is a polynomial with respect to z, i.e., $\bar{p}(z)\in\mathbb{R}[z]$. It is easy to see that *all the zeros of the rational function $p(z)$ lie in the open unit disc in the complex plane if and only if all the zeros of the polynomial $\bar{p}(z)$ are in the same disc.*

In the next section we shall show that

$$\bar{p}(z)=z^{n+t+m}-\left[p_0(z)+\bar{p}_1(z)\right],$$

where $p_0(z),\bar{p}_1(z)\in\mathbb{R}[z]$, $\deg p_0(z)\leqslant n+t-1$, $\bar{p}_1(z)=z^{n+t}p_1(z)$ for some

$p_1(z) \in \mathbb{R}[z]$, $\deg p_1(z) \leqslant m-1$. Moreover, if $m \in \mathbb{N}$, we have $p_1(z) = a_{n-m+1} + za_{n-m+2} + \cdots + z^{m-1}a_n$, whereas all coefficients of $p_0(z)$ can be chosen arbitrary by an appropriate selection of $\{f_j\}_{j=0}^{t}$, $f_j \in \mathbb{R}^{1\times n}$. Accordingly, we formulate the following:

PROBLEM B. For given numbers $n \in \mathbb{N}$, $m \in \mathbb{N}_0$ and a polynomial $p_1(z) \in \mathbb{R}[z]$ such that $\deg p_1(z) \leqslant m-1$, find $t \in \mathbb{N}_0$ and a polynomial $p_0(z) \in \mathbb{R}[z]$, $\deg p_0(z) \leqslant n+t-1$, so that all the zeros of the polynomial

$$\bar{p}(z) \triangleq z^{n+t+m} - \left[p_0(z) + z^{n+t}p_1(z)\right]$$

lie in the open unit disc in the complex plane.

The problem as above is obviously equivalent to the original one, so instead of Problem A we shall only consider Problem B. In the next section we shall make use of Newton's formulae relating coefficients of a given polynomial to sums of the rth powers of its zeros in order to show the last problem can be reduced to the following:

PROBLEM C. For given numbers $n \in \mathbb{N}$, $m \in \mathbb{N}_0$, given a sequence $\{w_r\}_{r=1}^{m}$ of m real numbers, find $t \in \mathbb{N}_0$ and a symmetric (with respect to the real axis) sequence S of $n+m+t$ complex numbers, $S = \{s_j\}_{j=1}^{n+m+t}$, so that all s_j lie in the open unit disc in the complex plane and for all $r = 1, 2, \ldots, m$ the equality

$$\sum_{j=1}^{n+m+t} s_j^r = w_r$$

holds.

In Section 3, the device of complete induction with respect to the number $m \in \mathbb{N}_0$ will be of use for a proof that *we can always find a desired sequence* S *with the properties defined by Problem* C. In this way we can find a solution of Problem C, which means that *Problem* B *and our original Problem* A *have solutions*.

The paper is concluded with Section 4, where an application of the theory to the stabilizability problem for a linear system with a delayed control action will be given.

It is reasonable to say that Problems A and B are spectral-synthesis problems.

2. PRELIMINARY RESULTS

Let $\bar{p}(z) \in \mathbb{R}[z]$ be defined as in the previous section. Hence

$$\bar{p}(z) \triangleq z^{m+t}p(z) = z^{m+t}\{z^n - [\tilde{a}_1(z^{-1}) + z\tilde{a}_2(z^{-1}) + \cdots + z^{n-1}\tilde{a}_n(z^{-1})]\},$$

where, for $k = 1,2,\dots,n$, we have $\tilde{a}_k(z^{-1}) = a_k + \sum_{j=0}^{t}(z^{-1})^{m+j}f_{jk}$. Substituting $\tilde{a}_k(z^{-1})$ in $\bar{p}(z)$ and rearranging terms, we obtain

$$\begin{aligned}\bar{p}(z) &= z^{n+t+m} - \sum_{k=1}^{n} z^{k-1}\left(z^{m+t}a_k + \sum_{j=0}^{t} z^{t-j}f_{jk}\right)\\ &= z^{n+t+m} - [p_0(z) + z^{n+t}p_1(z)],\end{aligned}$$

where

$$p_0(z) \triangleq \sum_{k=1}^{n-m} z^{t+k-1+m}a_k + \sum_{k=1}^{n}\sum_{j=0}^{t} z^{t+k-1-j}f_{jk}$$

and

$$z^{n+t}p_1(z) \triangleq \sum_{k=n-m+1}^{n} z^{t+k-1+m}a_k.$$

After simple calculations we find that

$$p_1(z) = \sum_{i=-m+1}^{0} z^{i-1+m}a_{i+n},$$

i.e., $p_1(z) = 0$ if $m = 0$, and $p_1(z) = a_{n-m+1} + za_{n-m+2} + \cdots + z^{m-1}a_n$ if $m \in \mathbb{N}$. We consider now the polynomial $p_0(z)$. First, observe that $\deg p_0(z) \leqslant t + n - 1$. We have

LEMMA 1. *For any given polynomial* $g(z) \in \mathbb{R}[z]$, $\deg g(z) \leqslant t + n - 1$, *there exists a sequence*

$$\{f_{jk}\}_{j=0,1,\dots,t,\ k=1,2,\dots,n}$$

of real numbers such that $p_0(z) = g(z)$, where

$$p_0(z) \triangleq \sum_{k=1}^{n-m} z^{t+k-1+m} a_k + \sum_{k=1}^{n} \sum_{j=0}^{t} z^{t+k-1-j} f_{jk}$$

and $a_1, a_2, \ldots, a_{n-m} \in \mathbb{R}$ are given.

Proof. Observe that $\deg(\sum_{k=1}^{n-m} z^{t+k-1+m} a_k) \leqslant t+n-1$. For $j = 1,2,\ldots,t$ and $k = 2,3,\ldots,n$, let $f_{jk} = 0$. Then we have

$$\begin{aligned}\sum_{k=1}^{n} \sum_{j=0}^{t} z^{t+k-1-j} f_{jk} &= \sum_{j=0}^{t} z^{t-j} f_{j1} + \sum_{k=2}^{n} z^{t+k-1} f_{0k} \\ &= f_{t1} + z f_{t-1,1} + \cdots + z^t f_{01} + z^{t+1} f_{02} + z^{t+2} f_{03} \\ &\quad + \cdots + z^{t+n-1} f_{0n}.\end{aligned}$$

Now, it is easily seen that selecting appropriate real numbers $f_{t1}, f_{t-1,1}, \ldots, f_{01}, f_{02}, f_{03}, \ldots, f_{0n}$, we achieve the desired equality $p_0(z) = g(z)$. ∎

From the lemma above it follows directly that Problem A and Problem B, defined in Section 1, are equivalent in an obvious sense. Hence, we shall study Problem B. For this, we shall need some well-known results concerning symmetric polynomials; see e.g., [4, Chapter IX, § 2] and [6, Chapter I, Section 3].

Let $\mathbb{R}_s[x_1, x_2, \ldots, x_k]$ denote the ring of symmetric polynomials in $x_1, x_2, \ldots, x_k$ with coefficients in $\mathbb{R}$. The simplest elements of the ring $\mathbb{R}_s[x_1, x_2, \ldots, x_k]$ are the polynomials

$$d_r(x_1, x_2, \ldots, x_k) \triangleq x_1^r + x_2^r + \cdots + x_k^r.$$

Here $r \in \mathbb{N}$. The polynomial d_r is called the *sum of rth powers*.

Now, let $v(z) \in \mathbb{R}[z]$ be a polynomial of the form

$$v(z) = z^k - \left(v_0 + z v_1 + \cdots + z^{k-1} v_{k-1}\right).$$

Let $s_1, s_2, \ldots, s_k \in \mathbb{C}$ be the zeros of $v(z)$, so that

$$v(z) = (z - s_1)(z - s_2) \cdots (z - s_k).$$

The relations

$$\begin{aligned}
d_1(s_1, s_2, \ldots, s_k) &= v_{k-1},\\
d_2(s_1, s_2, \ldots, s_k) &= 2v_{k-2} + v_{k-1}d_1(s_1, s_2, \ldots, s_k),\\
&\vdots\\
d_i(s_1, s_2, \ldots, s_k) &= iv_{k-i} + v_{k-i+1}d_1(s_1, s_2, \ldots, s_k) + \cdots\\
&\quad + v_{i-2}d_{i-2}(s_1, s_2, \ldots, s_k) + v_{i-1}d_{i-1}(s_1, s_2, \ldots, s_k),\\
&\vdots\\
d_k(s_1, s_2, \ldots, s_k) &= kv_0 + v_1 d_1(s_1, s_2, \ldots, s_k) + \cdots\\
&\quad + v_{k-2}d_{k-2}(s_1, s_2, \ldots, s_k) + v_{k-1}d_{k-1}(s_1, s_2, \ldots, s_k)
\end{aligned}$$

are well known as *Newton's formulae*. These formulae give a direct correspondence between the coefficients $v_0, v_1, \ldots, v_{k-1}$ of the polynomial $v(z)$ and sums of rth powers of its zeros $s_1, s_2, \ldots, s_k$. It is important to note that, for every $i = 1, 2, \ldots, k$, the polynomials $d_1(s_1, s_2, \ldots, s_k), d_2(s_1, s_2, \ldots, s_k), \ldots, d_i(s_1, s_2, \ldots, s_k)$ are uniquely defined by $v_{k-1}, v_{k-2}, \ldots, v_{k-i}$. Conversely, the coefficients $v_{k-1}, v_{k-2}, \ldots, v_{k-i}$ of $v(z)$ are uniquely determined by $d_1(s_1, s_2, \ldots, s_k), d_2(s_1, s_2, \ldots, s_k), \ldots, d_i(s_1, s_2, \ldots, s_k)$.

Let us consider Problem B. We have

$$\begin{aligned}
\bar{p}(z) = z^{n+t+m} - \big(&c_0 + zc_1 + \cdots + z^{n+t-1}c_{n+t-1}\\
&+ z^{n+t}a_{n-m+1} + z^{n+t+1}a_{n-m+2} + \cdots + z^{n+t+m-1}a_n\big)
\end{aligned}$$

where $n \in \mathbb{N}$, $m \in \mathbb{N}_0$, and the real numbers $a_{n-m+1}, a_{n-m+2}, \ldots, a_n$ are given. We want to find a number $t \in \mathbb{N}_0$ and real numbers $c_0, c_1, \ldots, c_{n+t-1}$ such that for every $s \in \mathbb{C}$, $\bar{p}(s) = 0$ implies $|z| < 1$. Let $S = \{s_j\}_{j=1}^{n+m+t}$ be a symmetric (with respect to the real axis) sequence of $n + m + t$ complex numbers which represent all the zeros of the (still unspecified) polynomial

$\bar{p}(z)$. Hence, by Newton's formulae some constraints on S are imposed. More precisely, these constraints can be expressed as the following equalities:

$$d_1(s_1, s_2, \dots, s_{n+m+t}) = w_1,$$

$$d_2(s_1, s_2, \dots, s_{n+m+t}) = w_2,$$

$$\vdots$$

$$d_m(s_1, s_2, \dots, s_{n+m+t}) = w_m.$$

Of course, $w_1, w_2, \dots, w_m$ are real numbers uniquely defined (with the aid of Newton's formulae) by the given real numbers $a_{n-m+1}, a_{n-m+2}, \dots, a_n$. Conversely, $w_1, w_2, \dots, w_m$ define uniquely appropriate coefficients of the polynomial $\bar{p}(z)$.

Now it is an easy matter to verify that Problem B is equivalent to the following: For given numbers $n \in \mathbb{N}$, $m \in \mathbb{N}_0$, given a sequence $\{w_r\}_{r=1}^m$, $w_r \in \mathbb{R}$, find $t \in \mathbb{N}_0$ and a symmetric (with respect to the real axis) sequence $S = \{s_j\}_{j=1}^{n+m+t}$, $s_j \in \mathbb{C}$, so that for every $j = 1,2,\dots,n+m+t$, $|s_j| < 1$, and for every $r = 1,2,\dots,m$, $d_r(s_1, s_2, \dots, s_{n+m+t}) = w_r$. By the way, we have shown that instead of Problem B (or A) we may concentrate on Problem C defined in Section 1.

3. PROBLEM C HAS A SOLUTION

We shall use induction on m. For $m = 0$, $\{w_r\}_{r=1}^0$ is interpreted as the empty sequence, and it means that there is *no* constraint, of the form $d_r(s_1, s_2, \dots, s_{n+m+t}) = w_r$, on S. Hence we may take (for example) $t = 0$ and $s_1 = s_2 = \cdots = s_n = 0$. Now, we may assume that Problem C has a solution for some natural number m_0. Thus, for given $n \in \mathbb{N}$ and any given sequence $\{w_r\}_{r=1}^{m_0}$, $w_r \in \mathbb{R}$, we can find a number $t_0 \in \mathbb{N}_0$ and a symmetric (with respect to the real axis) sequence $S_0 = \{s_{0i}\}_{i=1}^{n+m_0+t_0}$, $s_{0i} \in \mathbb{C}$, so that for every $i = 1,2,\dots,n+m_0+t_0$, $|s_{0i}| < 1$, and for every $r = 1,2,\dots,m_0$, $d_r(s_{01}, s_{02}, \dots, s_{0,n+m_0+t_0}) = w_r$.

Of special interest for the sequel is the following:

LEMMA 2. *Let $m_0 \in \mathbb{N}$, $w \in \mathbb{R}$ be given; $m \triangleq m_0 + 1$. There exist $h \in \mathbb{N}$ and a symmetric (with respect to the real axis) sequence* $S_1 = \{s_{1i}\}_{i=1}^h$,

$s_{1i} \in \mathbb{C}$, *so that for every* $i = 1,2,\ldots,h$, $|s_{1i}| < 1$ *and for every* $r = 1,2,\ldots,m_0$, $d_r(s_{11}, s_{12},\ldots,s_{1h}) = 0$, *whereas* $d_m(s_{11}, s_{12},\ldots,s_{1h}) = w$.

Proof. Let us consider an auxiliary polynomial

$$g(z) \triangleq z^m - g_0$$

where $g_0 \in \mathbb{R}$ will be specified later. Let $s_{21}, s_{22},\ldots,s_{2m}$ be all the zeros of the polynomial $g(z)$. Using Newton's formulae we get

$$d_1(s_{21}, s_{22},\ldots,s_{2m}) = 0,$$

$$d_2(s_{21}, s_{22},\ldots,s_{2m}) = 0,$$

$$\vdots$$

$$d_{m_0}(s_{21}, s_{22},\ldots,s_{2m}) = 0,$$

and

$$d_m(s_{21}, s_{22},\ldots,s_{2m}) = mg_0.$$

Let $k \triangleq \min\{j \in \mathbb{N} : |w/j| < m\}$, $g_0 \triangleq w/km$, and

$$\bar{g}(z) \triangleq [g(z)]^k = \left(z^m - \frac{w}{km}\right)^k.$$

Observe that $\bar{g}(s) = 0$ if and only if $s^m = w/km$, so that $|s| < 1$. Let $s_{1i} \triangleq s_{21}$ for $i = 1,2,\ldots,k$, $s_{1i} \triangleq s_{22}$ for $i = k+1, k+2,\ldots,2k,\ldots, s_{1i} \triangleq s_{2m}$ for $i = (m-1)k+1, (m-1)k+2,\ldots,mk$, $h \triangleq mk$, and $S_1 \triangleq \{s_{1i}\}_{i=1}^{mk}$. We see immediately that S_1 is symmetric (with respect to the real axis), and for every $i = 1,2,\ldots,mk$, $|s_{1i}| < 1$. Another simple observation is that for every $r = 1,2,\ldots,m_0, m_0+1$,

$$d_r(s_{11}, s_{12},\ldots,s_{1,mk}) = kd_r(s_{21}, s_{22},\ldots,s_{2m}).$$

But this means that the sequence S_1 has the properties desired. ■

Let m_0, t_0, and s_{0i} be defined by induction hypothesis. Let $m \triangleq m_0 + 1$ and $\{w_r\}_{r=1}^m$ be determined as for Problem C. Set $w \triangleq w_m - d_m(s_{01}, s_{02},\ldots,$

$s_{0,n+m_0+t_0}$). Let h and s_{1i} be given by Lemma 2. Now, let $t \triangleq n+m_0+t_0+h$, $s_i = s_{0i}$ for $i=1,2,\ldots,n+m_0+t_0$, and $s_{i+n+m_0+t_0} = s_{1i}$ for $i=1,2,\ldots,h$. From the properties of the sequences $S_0 = \{s_{0i}\}_{i=1}^{n+m_0+t_0}$ and $S_1 = \{s_{1i}\}_{i=1}^{h}$, and also from the definition of the sum of rth powers, we directly deduce that the number t just defined and the sequence $S \triangleq \{s_i\}_{i=1}^{t}$ serve as the required solution of Problem C. Thus we have proved the following:

THEOREM. *There exist a number $t \in \mathbb{N}_0$ and a sequence S with the properties defined by Problem C.*

The equivalence of Problems A, B, C gives also

COROLLARY. *Problems* A, B *and* C *have solutions.*

4. AN APPLICATION

Let us consider a difference equation

$$x_{k+1} = Ax_k + bu_{k-m}$$

where $k \in \mathbb{N}_0$, $A \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^{n\times 1}$, and $m \in \mathbb{N}_0$. The difference equation represents a *linear discrete-time system* for which the nonnegative integer m is a *time delay in control action.* Such discrete-time system arises [5] if we consider a continuous-time system of the form $\dot{x}(t) = Fx(t) + gu(t-mh)$ with the "controls" $u(\cdot)$ constant on the intervals $[kh,(k+1)h]$, $k \in \mathbb{N}_0$.

It is assumed that the pair (A,b) defining the discrete-time system considered is such that A is *cyclic* and b is a *cyclic generator* (with respect to A) of $\mathbb{R}^{n\times 1}$. In other words, *controllability* of the pair (A,b) is assumed; see e.g. [8].

The problem we are going to speak about is usually called the *stabilizability problem* (see e.g. [8]). We shall need the concept of *stability* for a system of the form

$$x_{k+1} = \sum_{j=0}^{t} A_j x_{k-j}$$

where $k \in \mathbb{N}_0$ and, for $j=0,1,\ldots,t$, $A_j \in \mathbb{R}^{n\times n}$. The discrete-time system above is defined to be *asymptotically stable* if for every $x_{-t}, x_{-t+1},\ldots,$ $x_{-1}, x_0 \in \mathbb{R}^n$ an appropriate solution of the equation $x_{k+1} = \sum_{j=0}^{t} A_j x_{k-j}$ has the property $\lim_{k\to\infty} x_k = 0$. It can be easily shown (see e.g. [2]) that *the system considered is asymptotically stable if and only if all the zeros of the rational function* $\det[zI - \sum_{j=0}^{t} z^{-(m+j)} A_j]$ *lie in the open unit disc in the complex plane.*

The stated problem is as follows:

PROBLEM D. For the system $x_{k+1} = Ax_k + bu_{k-m}$ [where $(A, b) \in \mathbb{R}^{n\times n} \times R^{n\times 1}$ is controllable and $m \in \mathbb{N}_0$ is given] find $t \in \mathbb{N}_0$ and $\{f_j\}_{j=0}^{t}, f_j \in \mathbb{R}^{1\times n}$, so that the "closed loop" system

$$x_{k+1} = Ax_k + \sum_{j=0}^{t} bf_j x_{k-m-j}$$

is asymptotically stable.

If $m = 0$, the problem above is well known (see e.g. [1], [3], [7], [8]). For $m = 1$ it has been solved by the author [5]. Taking into account the formulation of Problem A, the Corollary, and conditions for asymptotic stability, we have

PROPOSITION. *There exist a number* $t \in \mathbb{N}_0$ *and a sequence* $\{f_j\}_{j=0}^{t}$, $f_j \in \mathbb{R}^{1\times n}$, *such that the "closed loop" system defined by Problem* D *is asymptotically stable.*

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REFERENCES

1 A. Feintuch, One-dimensional perturbations of cyclic operators, *Linear Algebra Appl.* 18:171–174 (1977).

2 R. E. Kalman and J. E. Bertram, Control system analysis and design via the "second method" of Lyapunov, II. Discrete-time systems, *ASME J. Basic Engr. Ser. D* 82:394–400 (1960).

3 R. E. Kalman, *Lectures on Controllability and Observability, C.I.M.E., Bologna*, 1968.

4 A. Mostowski and M. Stark, *Introduction to Higher Algebra*, Pergamon, Oxford, 1964.

5 K. M. Przyłuski, Stabilizability of the system $\dot{x}(t) = Fx(t) + Gu(t-h)$ by a discrete feedback control, *IEEE Trans. Automat. Control* 22:269–270 (1977).

6 H. H. Rosenbrock, *State-Space and Multivariable Theory*, Wiley-Interscience, New York, 1970.
7 J. C. Willems and S. K. Mitter, Controllability, observability, pole allocation, and state reconstruction, *IEEE Trans. Automat. Control* 16:582–595 (1971).
8 W. M. Wonham, *Linear Multivariable Control. A Geometric Approach*, Springer, Berlin, 1974.

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